

Future Oscillations around Phantom Divide in $f(R)$ Gravity

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Abstract

It is known that scalar-tensor theory of gravity admits regular crossing of the phantom divide line $w_{\text{DE}} = -1$ for dark energy, and existing viable models of present dark energy for its particular case – $f(R)$ gravity – possess one such crossing in the recent past, after the end of the matter dominated stage. It was recently noted that during the future evolution of these models the dark energy equation of state w_{DE} may oscillate with an arbitrary number of phantom divide crossings. In this paper we prove that the number of crossings can be infinite, present an analytical condition for the existence of this effect and investigate it numerically. With the increase of the present mass of the scalaron (a scalar particle appearing in $f(R)$ gravity) beyond the boundary of the appearance of such oscillations, their amplitude is shown to decrease very fast. As a result, the effect quickly becomes small and its beginning is shifted to the remote future.

I. INTRODUCTION

The accelerating expansion of the present Universe is confirmed by current precise observational data such as type Ia supernovae [1, 2], anisotropy of cosmic microwave background [3], large scale structure [4] and baryon acoustic oscillations [5, 6]. The standard cosmological constant (Λ)-Cold-Dark-Matter (CDM) model is indeed able to explain these observational results within observational errors. In this model a cosmological constant is regarded as a new fundamental physical constant. However, the required value of the cosmological constant is very tiny compared with any known physical scales. Thus, its relation to the standard quantum theory of known particles and fields is not understood today although some nonperturbative effects may naturally generate such a small quantity, see e.g. Refs. [7, 8]. More generally, a source of the current cosmic acceleration is called dark energy (DE). Further we shall use the more detailed term "present DE" for it to distinguish it from primordial DE which was responsible for another accelerated expansion regime, dubbed inflation, which occurred in the very early Universe [9–11]. The relation of primordial DE to the known elementary particles has not been established, too.

Both primordial and present DE can have either a physical origin (some new physical fields of matter) or a geometrical one. In the latter case, the Einstein gravity becomes modified. One of the simplest and self-consistent generalizations of the Einstein gravity is $f(R)$ gravity which incorporates a new phenomenological function $f(R)$ of the Ricci scalar R (with d^2f/dR^2 not identically zero) into the action, see Eq. (3) below. For a long time this theory of gravity was known to contain viable inflationary models, among them the simplest one introduced in Ref. [9] that remains in agreement with the most recent observational data. Thus, $f(R)$ gravity can successively describe primordial DE. Rather recently, after many unsuccessful trials, viable models of present dark energy were found [12–14] which provide non-trivial alternatives to the standard Λ CDM model.¹ This theory is a special class of the scalar-tensor theory of gravity with the vanishing Brans-Dicke parameter ω_{BD} . It contains a new scalar degree of freedom dubbed "scalaron" in Ref. [9], thus, it is a

¹ In order not to destroy the standard early Universe cosmology, including the recombination, the correct Big Bang nucleosynthesis and inflation of any kind, these models of present DE possessing a non-trivial form of $f(R)$ in the low- R , $R > 0$ region have to be further generalized by changing the behaviour of $f(R)$ at large R and by extending it to the region of negative R , see Ref. [15]. However, this generalization is not important for our study.

nonperturbative generalization of the Einstein gravity. From the quantum point of view, scalaron is a massive spin-0 particle which mass depends on R . We consider $f(R)$ gravity as a phenomenological macroscopic theory of gravity, alternative to the Einstein one, without discussing its microscopic origin.²

The existence of this additional degree of freedom imposes a number of constraints on the functional form of $f(R)$ in viable cosmological models. In particular, in order to have the correct Newtonian limit, as well as the standard matter-dominated stage with the scale factor behaviour $a(t) \propto t^{2/3}$ driven by cold dark matter and baryons, the following conditions should be fulfilled for $R \gg R_0$ where $R_0 \equiv R(t_0) \sim H_0^2$, t_0 is the present moment and H_0 is the Hubble constant, and up to curvatures in the centre of neutron stars:

$$|f(R) - R| \ll R, \quad |f'(R) - 1| \ll 1, \quad Rf''(R) \ll 1. \quad (1)$$

Here the prime denotes the derivative with respect to the argument R . Furthermore, $f(R)$ should satisfy the following conditions to guarantee both that Newtonian gravity solutions are stable and that the standard matter-dominated Friedmann-Robertson-Walker (FRW) stage remains an attractor with respect to an open set of neighbouring generic cosmological solutions in $f(R)$ gravity:

$$f'(R) > 0, \quad f''(R) > 0. \quad (2)$$

In quantum language, the first condition means that gravity is attractive and graviton is not a ghost, while the second one – that scalaron is not a tachyon. Specific functional forms of $f(R)$ satisfying these conditions, as well as laboratory and Solar system tests of gravity, and possessing a future stable (or at least metastable) de Sitter stage that is required for correct description of observable properties of present DE, have been proposed in Refs. [12–14], and much work has been carried out on their cosmological consequences.

In order to describe the difference between FRW background solutions of $f(R)$ gravity and the Λ CDM model, it is useful to introduce the effective equation-of-state (EoS) parameter for DE $w_{\text{DE}} \equiv P_{\text{DE}}/\rho_{\text{DE}}$ where the effective pressure P_{DE} and the effective energy

² Note the simplest possible mechanism that has attracted a new interest recently: a scalar field ϕ with some potential and the non-minimal coupling $-\xi R\phi^2/2$ to the Einstein gravity in the limit of a very large negative ξ (i.e. the sign of coupling is opposite to that of the conformally coupled case). However, this mechanism leads to $df/dR > 1$. So, while sufficient to produce the functional form of $f(R)$ needed for successful inflationary models, it is not useful for construction of viable models of present DE.

density ρ_{DE} of DE are determined using the Einsteinian representation of gravitational field equations, see Eqs. (9), (10) below. Another independent parameter which describes scalar (density) perturbations on a FRW background is the gravitational growth index γ defined as $d \ln \delta / d \ln a \equiv \Omega_m(z)^{\gamma(z)}$ where $\delta \equiv \delta \rho_m / \rho_m$ and $\Omega_m \equiv 8\pi G \rho_m / 3H^2$ are a matter density fluctuation and the density parameter for matter, respectively. In $f(R)$ gravity, w_{DE} is time dependent and γ is time and scale dependent whilst they keep the constant value $w_{\text{DE}} = -1$ and $\gamma \approx 6/11$ in the Λ CDM model. Time and scale dependency of γ generate an additional transfer function for matter density fluctuation that constrains the model parameter region [16–18].

One of the most interesting features of geometrical DE distinguishing it from physical DE based on non-ghost physical fields minimally coupled to gravity, like quintessence, is the possibility of phantom behaviour, $w_{\text{DE}} < -1$, of DE. Moreover, this behaviour may well be temporary with DE becoming normal, $w_{\text{DE}} > -1$, after smooth crossing of the phantom boundary $w_{\text{DE}} = -1$. In particular, models of geometrical DE based on scalar-tensor gravity were long known to admit this property [19]. $f(R)$ gravity is a particular case of scalar-tensor gravity, so it permits phantom behaviour of DE and smooth crossing of the phantom boundary, too. Existing observational data do not exclude the possibility of phantom behaviour of DE (though they do not specifically favour it, too) for the following simple reason: as was noted above, DE in the particular form of an exact cosmological constant Λ is in a good agreement with all data. But since $w_\Lambda \equiv -1$, it lies exactly at the phantom boundary. Thus, any small deviation of DE from Λ to the direction of decreasing w_{DE} results in its phantom behaviour. So, theory has to be prepared for this possibility that explains large interest in DE models admitting it. However, it is clear already that this "phantomness" should be small. In particular, if it is assumed for simplicity that $w_{\text{DE}} = \text{const}$, when $|w_{\text{DE}} + 1| < 0.1$ at the approximately 2σ confidence level [3].

Moreover, viable $f(R)$ models of present DE [12–14] generically exhibit phantom behaviour during the matter-dominated stage and one recent crossing of the phantom divide $w_{\text{DE}} = -1$ even in the case of the smoothest behaviour of a FRW scale factor $a(t)$, when there were no superimposed small oscillations of $a(t)$ in the past (in quantum language, no condensate of primordial scalarons with the zero momentum) [12, 15, 16, 20]. From the physical point of view, the absence of primordial scalarons in the viable $f(R)$ models of present DE is needed in order not to destroy the standard early Universe cosmology and

it can be achieved by primordial inflation of any kind, see Ref. [15] for a detailed discussion. Using numerical calculations, it has been recently shown that even in this smoothest case the EoS parameter w_{DE} can oscillate around the future de Sitter solution in these DE models [21], see also Ref. [22].

To investigate the phenomenon of multiple crossing of the phantom divide in more detail and analytically, in the present paper we prove that this crossing can indeed occur infinitely many times during the future evolution of viable $f(R)$ models of present DE if the scalaron mass at a future stable de Sitter stage in these models is sufficiently large. Though this phenomenon is not directly observable since it refers to remote future, it is interesting and important from the theoretical point of view. Also, it is possible to check from observational data at the present moment if the derived analytical criterion for the existence of the infinite number of crossings is satisfied or not.

Thus, the present paper focuses on the oscillatory behaviour of w_{DE} around the phantom divide $w_{\text{DE}} = -1$ in the future. In Sec. II, we review the stability conditions and the condition of the existence of a stable future de Sitter stage in $f(R)$ gravity, and derive the condition for the existence of the infinite number of these oscillations analytically using the perturbation theory around the de Sitter solution. In Sec. III, we focus on the specific viable model of present DE in $f(R)$ gravity and present results from numerical calculations relating this condition to observable properties of the Universe at the present time. Sec. IV is devoted to conclusions and discussion.

II. THE CRITERIA

The action studied is of the form

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_m, \quad (3)$$

where $f(R)$ is a function of the Ricci scalar R and S_m denotes matter action with the minimal coupling to gravity. Field equations are derived as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + T_{\mu\nu}^{\text{DE}}), \quad (4)$$

$$8\pi G T_{\mu\nu}^{\text{DE}} = (1 - F)R_{\mu\nu} - \frac{1}{2}(R - f)g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)F \quad (5)$$

where $F = df/dR$. We use the representation (4),(5) to define the effective energy-momentum tensor $T_{\mu\nu}^{\text{DE}}$ of DE. The $(0,0)$ and (i,i) components of the field equations are

$$3FH^2 = \frac{RF - f}{2} - 3H\dot{F} + 8\pi G\rho, \quad (6)$$

$$6F\frac{\ddot{a}}{a} = RF - f - 3(\ddot{F} + H\dot{F}) - 8\pi G(\rho + 3P). \quad (7)$$

It is also useful to use the trace equation:

$$RF - 2f + 3\Box F = 8\pi GT. \quad (8)$$

Thus, the effective energy density, the pressure and the EoS parameter of DE have the form:

$$8\pi G\rho_{\text{DE}} \equiv 3H^2 - 8\pi G\rho = -3(1-F)\frac{\ddot{a}}{a} + \frac{R-f}{2} - 3H\dot{F}, \quad (9)$$

$$8\pi GP_{\text{DE}} \equiv -2\dot{H} - 3H^2 - 8\pi GP = (1-F)\left(\frac{\ddot{a}}{a} + 2H^2\right) - \frac{R-f}{2} + \ddot{F} + 2H\dot{F}, \quad (10)$$

$$w_{\text{DE}} + 1 = \frac{2(1-F)(-\ddot{a}/a + H^2) + \ddot{F} - H\dot{F}}{-3(1-F)\ddot{a}/a + (R-f)/2 - 3H\dot{F}}. \quad (11)$$

During an asymptotic de Sitter regime, the matter density decreases rapidly as $\rho \propto e^{-3H_1 t}$ and soon can be neglected. Therefore, it follows from Eq. (8) that a constant value of the Ricci scalar $R = R_1 = \text{const} = 12H_1^2$ at a de Sitter regime is given by a root of the algebraic equation

$$2f_1 = R_1 F_1 \quad (12)$$

where $f_1 \equiv f(R_1)$ and $F_1 \equiv F(R_1)$. At the de Sitter regime, DE is characterized by $8\pi G\rho_{\text{DE},1} = -8\pi GP_{\text{DE},1} = \frac{R_1}{4}$, thus $w_{\text{DE},1} = -1$.

To investigate the stability of the future de Sitter solution and the possibility of oscillatory behaviour around it, we expand Eqs. (6), (8) in the perturbation series with respect to $\delta R \equiv R - R_1$ and $\delta H \equiv H - H_1$. In the first order in δR and δH ,

$$\delta H = -\frac{H_1 F_{R1}}{2F_1}(\delta R' - \delta R) + \frac{1}{2F_1 H_1} \frac{8\pi G\rho_m}{3}, \quad (13)$$

$$\delta R'' + 3\delta R' + \frac{1}{3H_1^2} \left(\frac{F_1}{F_{R1}} - R_1 \right) \delta R = \frac{8\pi G\rho_m}{3F_{R1}H_1^2}, \quad (14)$$

where prime denotes the derivative with respect to the number of e -folds $N \equiv \ln a = -\ln(1+z)$ and $F_{R1} \equiv F_R(R_1) \equiv dF(R_1)/dR$. Although the matter density term in the right-hand side has the zero order, we include it because $\rho_m \propto e^{-3H_1 t}$ is much smaller than background quantities at the future de Sitter stage.

Eq. (14) is solved as a sum of the homogeneous solution of Eq. (14) with the zero right-hand side, δR_{osc} , and the special solution of the non-homogeneous equation δR_{dec} :

$$\delta R = \delta R_{\text{dec}} + \delta R_{\text{osc}}. \quad (15)$$

Since $\rho_m = \rho_{m0}e^{-3N}$, δR_{dec} is obtained as

$$\delta R_{\text{dec}} = \frac{8\pi G \rho_{m0}}{F_1 - R_1 F_{R1}} e^{-3N}. \quad (16)$$

Notice that it describes a monotonically decaying mode.

On the other hand, the homogeneous solution δR_{osc} may have decaying, growing and oscillatory behaviour. In the case of monotonic behaviour (both roots of the characteristic equation for Eq. (14) are real), to keep the future de Sitter solution stable (a stable node), the coefficient of the third term in the left-hand side of Eq. (14) should be positive. So, the stability condition is [23]:

$$\frac{F_1}{F_{R1}} > R_1. \quad (17)$$

The oscillatory behaviour occurs when the de Sitter asymptote is a focus (complex roots). For this, the discriminant of the characteristic equation should be negative:

$$\frac{F_1}{F_{R1}} > \frac{25}{16} R_1. \quad (18)$$

Since the coefficient of the second term in the left-hand side of Eq. (14) is positive, the focus is always stable and the inequality (18) is stronger than (17). The criterion (18) of the oscillatory approach to the future de Sitter asymptote is equivalent to the condition

$$M_1^2 \equiv \frac{F_1 - R_1 F_{R1}}{3F_{R1}} > \frac{9H_1^2}{4} = \frac{3R_1}{16}, \quad (19)$$

where H_1 , R_1 and M_1 are the Hubble parameter, the scalar curvature and the scalaron mass at the future de Sitter state. If the oscillation condition is satisfied,

$$\delta R_{\text{osc}} = A e^{-3N/2} \sin(\omega N + \phi) \quad (20)$$

where $\omega \equiv 2\sqrt{\frac{F_1}{R_1 F_{R1}} - \frac{25}{16}}$, and A and ϕ are integration constants.

The perturbation of the EoS parameter $\delta w_{\text{DE}} = (\delta P_{\text{DE}} + \delta \rho_{\text{DE}})/\rho_{\text{DE},1}$ is calculated from $8\pi G(\rho_{\text{DE}} + P_{\text{DE}}) = -2\dot{H} - 8\pi G\rho_m$ and Eq. (13),

$$\delta w_{\text{DE}} = \frac{4}{R_1} \left[-\frac{R_1 F_{R1}}{3F_1} \delta R' + \frac{1}{3} \left(\frac{R_1 F_{R1}}{F_1} - 1 \right) \delta R + \left(\frac{4}{F_1} - 3 \right) \frac{8\pi G \rho_m}{3} \right]. \quad (21)$$

We decompose $\delta w_{\text{DE}} \equiv \delta w_{\text{dec}} + \delta w_{\text{osc}}$ as

$$\delta w_{\text{dec}} = \frac{4}{R_1} \left(\frac{1}{F_1 - R_1 F_{R1}} - 1 \right) 8\pi G \rho_{m0} (1+z)^3 \quad (22)$$

$$\delta w_{\text{osc}} = A(1+z)^{3/2} \frac{4}{R_1} \left[-\frac{R_1 F_{R1}}{3F_1} \omega \cos(\omega N + \phi) + \frac{1}{3} \left(\frac{5R_1 F_{R1}}{2F_1} - 1 \right) \sin(\omega N + \phi) \right]. \quad (23)$$

Thus, in the latter case of a stable de Sitter solution with oscillations, w_{DE} crosses the phantom boundary $w_{\text{DE}} = -1$ infinitely many times during the future evolution of the Universe.

III. THE SPECIFIC MODEL

We consider the following viable $f(R)$ model [14]:

$$f(R) = R + \lambda R_s \left[\left(1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right], \quad (24)$$

where n and λ are model parameters, and R_s is determined by the present observational data, namely, the ratio R_s/H_0^2 is well fit by a simple power-law $R_s/H_0^2 = c_n \lambda^{-p_n}$ with $(n, c_n, p_n) = (2, 4.16, 0.953)$, $(3, 4.12, 0.837)$, and $(4, 4.74, 0.702)$, respectively [16].

From Eq. (12), the equation for de Sitter solutions is

$$\alpha(r) \equiv r + 2\lambda \left[\frac{1 + (n+1)r^2}{(1+r^2)^{n+1}} - 1 \right] = 0, \quad (25)$$

where $r \equiv R_1/R_s$. It is obvious that the Minkowski space-time, $r = 0$, is one of the solutions. We denote the other positive solutions for $\alpha(r) = 0$ as $r_a \equiv R_{1a}/R_s$ and $r_b \equiv R_{1b}/R_s$. r_a and r_b can be estimated by considering the limits of small and large r . For $r \ll 1$, $\alpha(r) \simeq r[1 - \lambda n(n+1)r^3]$, and for $r \gg 1$, $\alpha(r) \simeq r - 2\lambda$. Therefore, the de Sitter solutions are $r = r_a \simeq [\lambda n(n+1)]^{-1/3}$ and $r = r_b \simeq 2\lambda$. Strictly speaking, these approximations are valid either for $\lambda \gg 1$ or, in the case of r_a , for $n \gg 1$ (while λ may be of the order of unity). However, it follows from the numerical analysis that the solutions for $n = 2$ and $\lambda = 3$ are already close enough to these analytical estimations.

One can check their stability and oscillatory behaviour using the stability parameter $\beta(r)$ and the oscillation parameter $\gamma(r)$ which are derived from Eqs. (17) and (18):

$$\beta(r) \equiv \frac{(1+r^2)[(1+r^2)^{n+1} - 2n\lambda r]}{2n\lambda[(2n+1)r^2 - 1]} - r > 0, \quad (26)$$

$$\gamma(r) \equiv \frac{(1+r^2)[(1+r^2)^{n+1} - 2n\lambda r]}{2n\lambda[(2n+1)r^2 - 1]} - \frac{25}{16}r > 0. \quad (27)$$

TABLE I: Stable de Sitter solutions for various model parameters n and λ . $r_b \equiv R_{1b}/R_s$ is a stable de Sitter solution in terms of the normalized Ricci scalar. β and γ denote the stability (26) and oscillation (27) parameters, respectively.

n	λ	r_b	$\beta(r_b)$	$\gamma(r_b)$
2	1	1.64	1.58	6.56×10^{-1}
2	3	5.99	8.54×10^2	8.51×10^2
2	10	20.0	3.23×10^5	3.23×10^5
3	1	1.94	1.37×10	1.26×10
3	3	6.00	1.53×10^4	1.53×10^4
3	10	20.0	6.17×10^7	6.17×10^7
4	1	1.99	5.07×10	4.95×10
4	3	6.00	3.31×10^5	3.31×10^5
4	10	20.0	1.44×10^{10}	1.44×10^{10}

Since $\gamma(r) = \beta(r) - 9r/16$, there is no oscillations for an unstable de Sitter solution, as it should be. From these criteria we see that $r = r_a$ and $r = r_b$ are an unstable de Sitter solution and a stable de Sitter solution, respectively. The specific values are presented in the Table I.

For a fixed n and various values of λ , we obtain λ_β and λ_γ as roots of $\beta(r_b) = 0$ and $\gamma(r_b) = 0$, respectively. Models are classified according to λ being in the intervals $\lambda < \lambda_\beta$, $\lambda_\beta < \lambda < \lambda_\gamma$, and $\lambda > \lambda_\gamma$, and in each region a de Sitter solution $r = r_b$ is unstable, stable without oscillations, and stable with oscillations, respectively. Although most of the parameters realize a stable de Sitter solution with oscillations (a stable focus), there exists a parameter region corresponding to a stable de Sitter solution without oscillation (a stable node). Fig. 1 suggests that such parameter regions are $0.944 < \lambda < 0.970$, $0.726 < \lambda < 0.744$ and $0.608 < \lambda < 0.622$ for $n = 2, 3$ and 4 , respectively.

We integrate the evolution equations numerically. Initial condition are set at $z = 10$ using the Λ CDM model, and the present moment is determined by the condition $\Omega_m = 0.27$. Fig. 2 shows that R approaches a stable de Sitter solution. It is seen from the right panel of Fig. 2 that the perturbation theory with respect to $\delta R \equiv R - R_{1b}$ is valid when $z \lesssim -0.8$ for $n = 2$, $\lambda = 1$, and when $z \lesssim -0.5$ for $n = 2$, $\lambda = 3$ or 10 . The oscillation of δR for

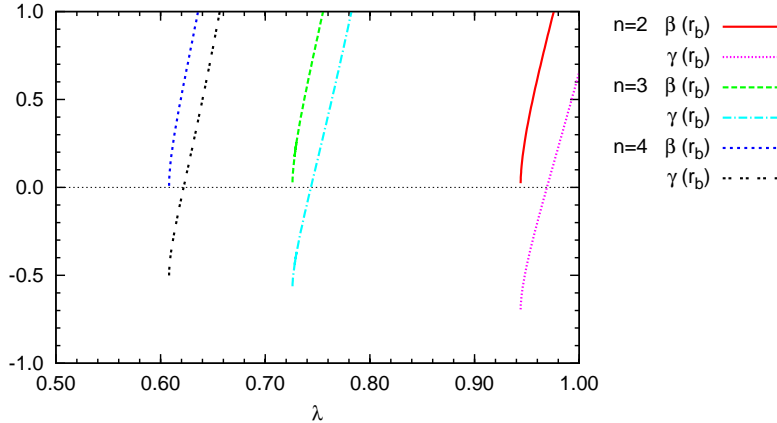


FIG. 1: Values of the stability parameter β and the oscillation parameter γ for stable de Sitter solutions for various model parameters. The parameter regions $\gamma(r_b) < 0 < \beta(r_b)$ and $\gamma(r_b) > 0$ correspond to stable de Sitter solutions without oscillations and with oscillations, respectively.

$n = 2$, $\lambda = 1$ is clearly seen. For $n = 2$ and $\lambda = 3$ or 10 , oscillations exist, too, but their amplitude is so small that we cannot see them. To make them visible, we have subtracted the decaying mode δR_{dec} in Fig. 3. The analytic solution for δR_{osc} fits the result well.

Fig. 4 depicts the evolution of the EoS parameter for $n = 2$ and $\lambda = 1, 3, 10$. The first phantom crossing occurred in the past at $z \sim 0.5$ in agreement with Ref. [16]. We subtract the decaying mode δw_{dec} and present the oscillation mode δw_{osc} in Fig. 5. The numerical results are very close to the analytic solutions for $n = 2$, $\lambda = 1$ and 3 . For $n = 2$, $\lambda = 10$, the amplitude of the oscillations is small and the frequency is large, so that we cannot distinguish them from numerical noise. Finally, we present the case $n = 2$, $\lambda = 0.95$ in Fig. 6 as an example of a non-oscillatory approach to the stable de Sitter solution. Note that the trajectories of δR and δw are convex upward and there is no oscillations indeed.

IV. CONCLUSIONS AND DISCUSSION

We have investigated conditions under which the effective EoS parameter w_{DE} of present DE in $f(R)$ gravity can oscillate an infinite number of times around the phantom boundary $w_{\text{DE}} = -1$ during the future evolution of the Universe. The analytical condition of the existence of this phenomenon, Eq. (18), is derived that depends on the properties of $f(R)$ near a future stable de Sitter stage only. The physical sense of this condition is that the

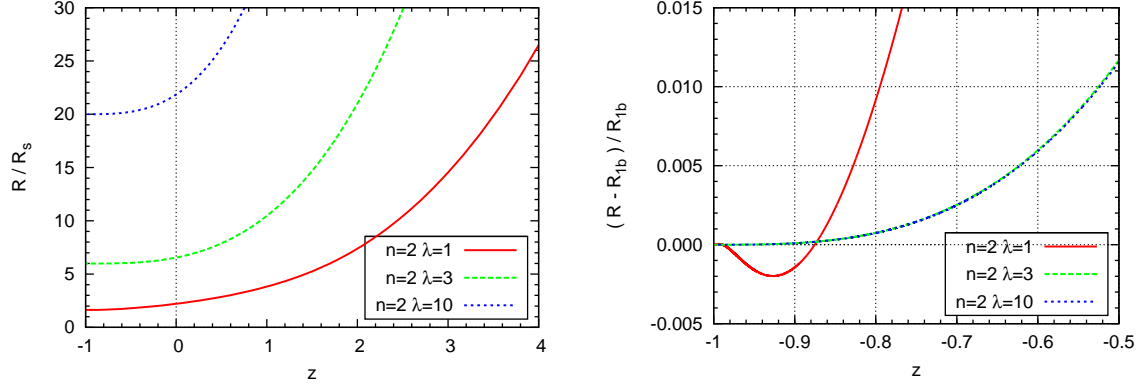


FIG. 2: Future evolution of the Ricci scalar. It approaches the stable de Sitter solution which is presented in the Table I.

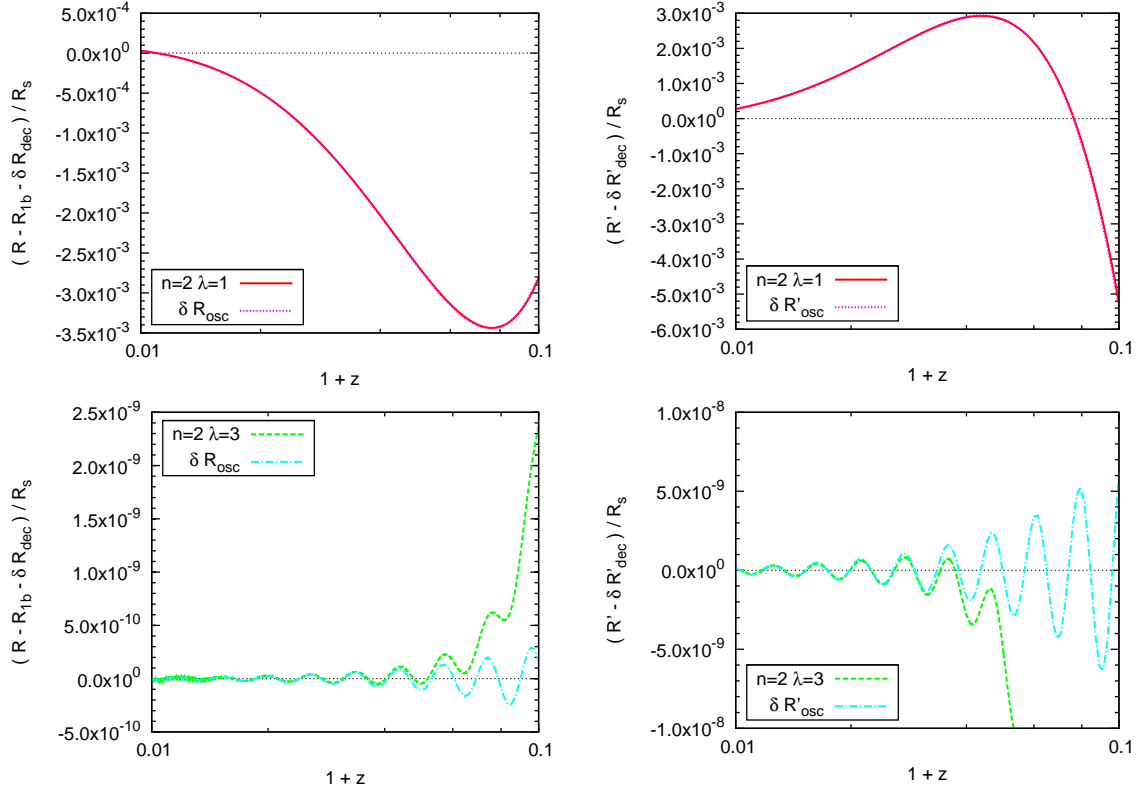


FIG. 3: Numerical results for $\delta R - \delta R_{dec}$ compared with the analytic solution for δR_{osc} .

rest mass of the scalaron (a massive scalar particle which arises in $f(R)$ gravity in addition to massless spin-2 graviton) should be sufficiently large at the future de Sitter stage. Thus, this phenomenon is generic. However, the amplitude of these oscillations has been shown to decrease fast with the increase of the scalaron mass beyond the boundary of the appearance

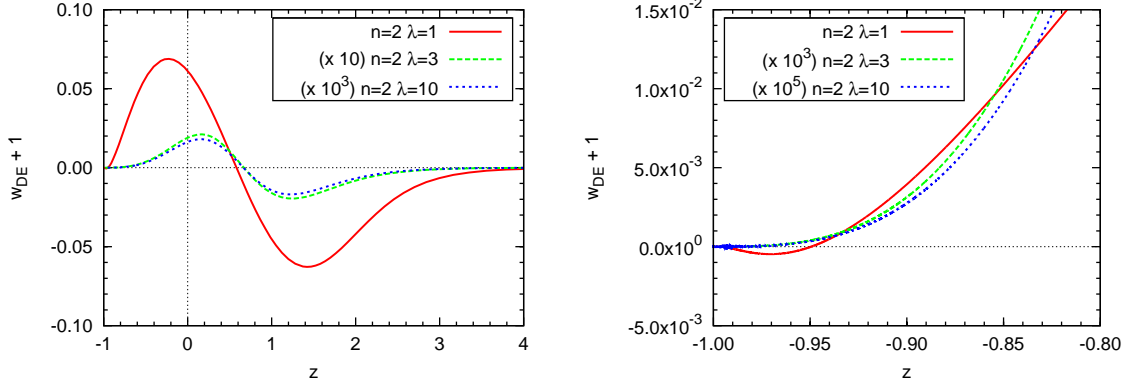


FIG. 4: Future evolution of the effective EoS parameter for dark energy.

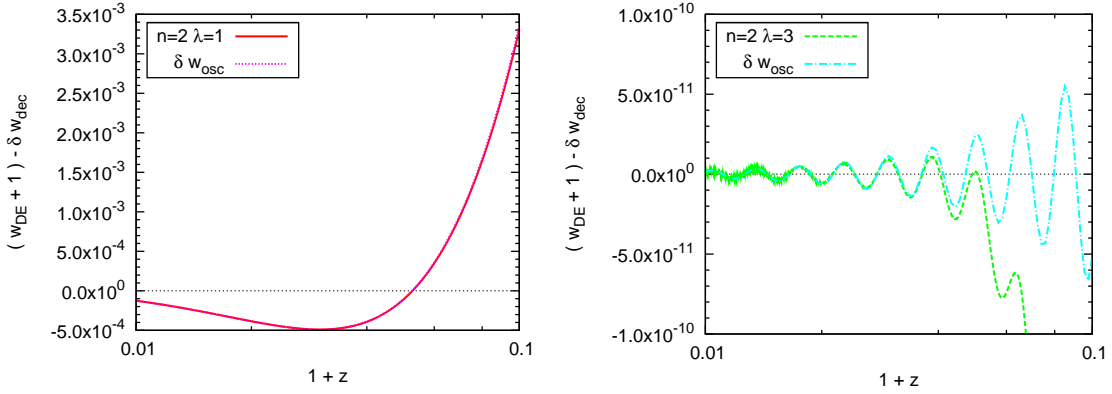


FIG. 5: Numerical results for $(1 + w_{\text{DE}}) - \delta w_{\text{dec}}$ compared with the analytic solution for δw_{osc} .

of such oscillations. As a result, the effect quickly becomes small and its beginning is shifted to the remote future. For real scalaron masses lying below this boundary, the future stable de Sitter stage is reached without the phantom boundary crossing. Analytic solutions for the behaviour of w_{DE} near the phantom boundary have been obtained in the first order of the small quantity $|w_{\text{DE}} + 1|$. Generically they have a monotonically decaying part δw_{dec} and a damped harmonic oscillatory part δw_{osc} . For a specific viable $f(R)$ model of present DE energy, numerical integration of FRW background evolution has been performed which future behaviour is in a good agreement with the analytic formulas.

All calculations have been done for the smoothest initial conditions in the past corresponding to the absence of a primordial homogeneous oscillating scalaron component. So, even in this case, an oscillating scalaron component (the condensate of scalarons with the zero momentum) arises around the present moment when the scalaron mass is comparable

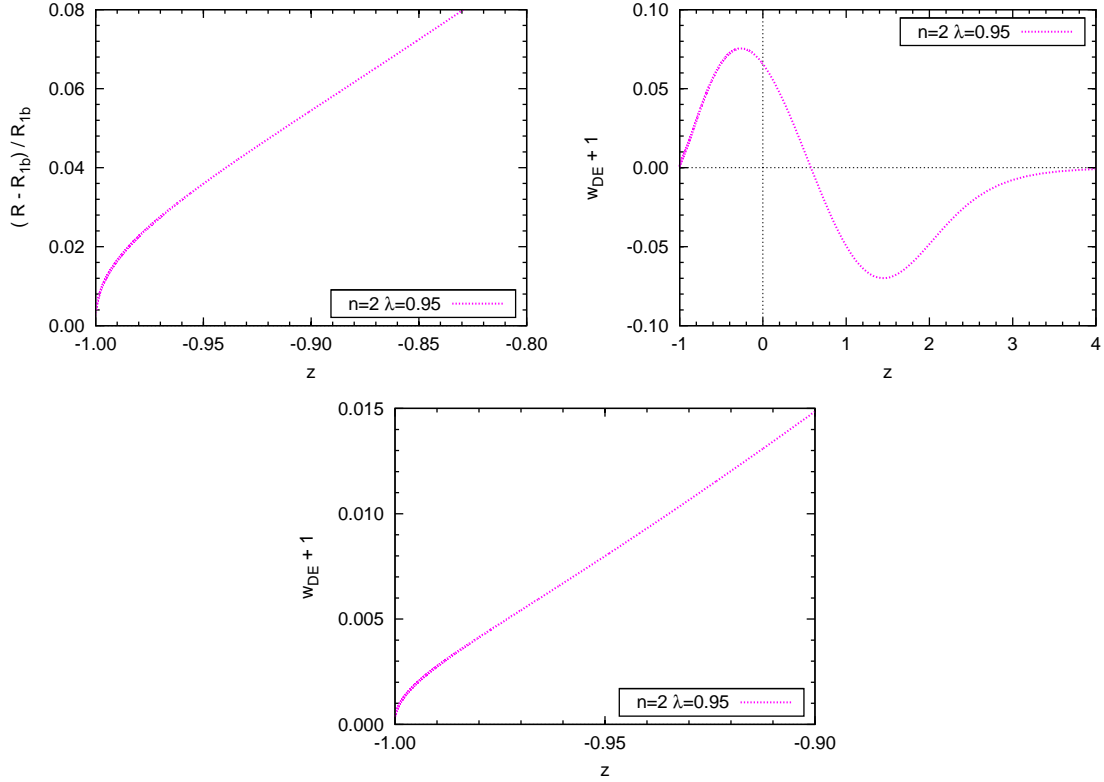


FIG. 6: Future evolution of the effective EoS parameter for dark energy for $n = 2$ and $\lambda = 0.95$. There is no oscillations.

to the Hubble constant H_0 (in the units where $\hbar = c = 1$), and it quickly becomes dominant over the non-relativistic matter component (cold dark matter and baryons) in the future. But its effective energy-momentum tensor in turn soon becomes negligible compared to an effective cosmological constant producing the future stable de Sitter stage. For less smooth initial conditions, more phantom boundary crossings may occur in the past. But these initial conditions are hardly compatible with the standard cosmology of the early Universe confirmed by numerous observational data. We hope to return to this question elsewhere.

Finally, though the very phenomenon of multiple (and even infinite) number of phantom boundary crossings in the future is not directly observable, it is very interesting and important from the theoretical point of view. Also, as follows from our numerical calculations of the full evolution from the remote past to the remote future, the scalaron mass at the future de Sitter stage is close to its present value. Therefore, in principle it is possible to check from observational data describing the present and the past of our Universe if the derived analytical criterion for the existence of an infinite number of oscillations in w_{DE} is satisfied

or not.

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- [1] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], *Astrophys. J.* **517**, 565 (1999) [arXiv:astro-ph/9812133].
 - [2] A. G. Riess *et al.* [Supernova Search Team Collaboration], *Astron. J.* **116**, 1009 (1998) [arXiv:astro-ph/9805201].
 - [3] E. Komatsu *et al.*, arXiv:1001.4538 [astro-ph.CO].
 - [4] M. Tegmark *et al.* [SDSS Collaboration], *Phys. Rev. D* **69**, 103501 (2004) [arXiv:astro-ph/0310723].
 - [5] D. J. Eisenstein *et al.* [SDSS Collaboration], *Astrophys. J.* **633**, 560 (2005) [arXiv:astro-ph/0501171].
 - [6] W. J. Percival, S. Cole, D. J. Eisenstein, R. C. Nichol, J. A. Peacock, A. C. Pope and A. S. Szalay, *Mon. Not. Roy. Astron. Soc.* **381**, 1053 (2007) [arXiv:0705.3323].
 - [7] J. Yokoyama, *Phys. Rev. Lett.* **88**, 151302 (2002) [arXiv:hep-th/0110137].
 - [8] C. Kiefer, F. Queisser and A. A. Starobinsky, arXiv:1010.5331.
 - [9] A. A. Starobinsky, *Phys. Lett. B* **91** (1980) 99.
 - [10] K. Sato, *Mon. Not. Roy. Astron. Soc.* **195**, 467 (1981).
 - [11] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981).
 - [12] W. Hu and I. Sawicki, *Phys. Rev. D* **76**, 064004 (2007) [arXiv:0705.1158].
 - [13] S. A. Appleby and R. A. Battye, *Phys. Lett. B* **654**, 7 (2007) [arXiv:0705.3199].

- [14] A. A. Starobinsky, JETP Lett. **86**, 157 (2007) [arXiv:0706.2041].
- [15] S. A. Appleby, R. A. Battye and A. A. Starobinsky, JCAP **1006**, 005 (2010) [arXiv:0909.1737].
- [16] H. Motohashi, A. A. Starobinsky and J. Yokoyama, Prog. Theor. Phys. **123**, 887 (2010) [arXiv:1002.1141].
- [17] H. Motohashi, A. A. Starobinsky and J. Yokoyama, Int. J. Mod. Phys. D **18**, 1731 (2009) [arXiv:0905.0730].
- [18] H. Motohashi, A. A. Starobinsky and J. Yokoyama, Prog. Theor. Phys. **124**, 541 (2010) [arXiv:1005.1171].
- [19] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000) [arXiv:gr-qc/0001066].
- [20] M. Martinelli, A. Melchiorri and L. Amendola, Phys. Rev. D **79**, 123516 (2009) [arXiv:0906.2350].
- [21] K. Bamba, C. Q. Geng and C. C. Lee, JCAP **1011**, 001 (2010) [arXiv:1007.0482].
- [22] H. W. Lee, K. Y. Kim and Y. S. Myung, Eur. Phys. J. **71**, 1585 (2011) [arXiv:1010.5556].
- [23] V. Müller, H.-J. Schmidt and A. A. Starobinsky, Phys. Lett. B **202**, 198 (1988).